

# Percolating level sets of the adjacency eigenvectors of $d$ -regular graphs

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**Abstract.** One of the most surprising discoveries in quantum chaos was that nodal domains of eigenfunctions of quantum-chaotic billiards and maps in the semi-classical limit display critical percolation. Here we extend these studies to the level sets of the adjacency eigenvectors of  $d$ -regular graphs. Numerical computations show that the statistics of the largest level sets (the maximal connected components of the graph for which the eigenvector exceeds a prescribed value) depend critically on the level. The critical level is a function of the eigenvalue and the degree  $d$ . To explain the observed behavior we study a random Gaussian waves ensemble over the  $d$ -regular tree. For this model, we prove the existence of a critical threshold. Using the local tree property of  $d$ -regular graphs, and assuming the (local) applicability of the random waves model, we can compute the critical percolation level and reproduce the numerical simulations. These results support the random-waves model for random regular graphs, suggested in [1] and provides an extension to Bogomolny's percolation model [2] for two-dimensional chaotic billiards.

## 1. Introduction

The statistics of the adjacency spectrum of random  $d$ -regular graphs (in the limit of large vertex number) displays the generic attributes associated with the spectra of quantum Hamiltonians (in the limit  $\hbar \rightarrow 0$ ) whose classical dynamics is chaotic. This observation

which was first based on numerical simulations [3], was further substantiated in [4, 5] where a link was made between the distribution of cycle counts in  $d$ -regular graphs, spectral statistics and Random Matrix Theory. In a way, the association of large  $d$ -regular graphs with chaotic dynamics is natural: The time evolution of random walks on a typical regular graph is known to mix exponentially fast [6, 7]. At the same time, the adjacency operator of a random regular graph is a symmetric matrix with identically (though not independently) distributed random variables (see section 2.2). Thus, it does not come as a surprise that in the limit of large graphs with a fixed degree, the spectral statistics of a typical regular graph should follow some of the universality classes related to systems with chaotic (or mixing) dynamics.

Recently, the study of level sets of eigenfunctions for two dimensional billiards and in particular the zero level sets (which are the nodal domains) have gained a considerable attention as possible indicators for the dynamics of the underlying classical system [8, 9, 2, 10, 11, 12]. The distribution of the (normalized) number of nodal domains in two dimensional billiards was examined in [9]. For chaotic billiards, this distribution was observed to converge in the semi classical limit into a universal measure, independent of the idiosyncratic dynamical characteristics of the billiard. These findings found an intriguing explanation in [2], where it was suggested that the distribution of nodal domains can be approximated by a non-correlated percolation process. Further connections with percolation theory were discovered in [11, 12] who showed that the boundary of the percolating domain reveal the  $SLE_6$  statistics. These observations motivated the research reported in the present paper, where we examine the distribution of the adjacency eigenvectors for typical regular graphs. In particular we investigate the morphology of the associated *level sets*: Given a graph  $G$  and a real function  $f(G)$  which is defined on the vertices of  $G$ , the  $\alpha$ -level sets of  $f$  in  $G$  are the maximal connected components of  $G$ , on which  $f$  exceeds the value  $\alpha$ .

Level sets of a special interest are the zero level sets - the nodal domains. In a previous paper [1], the dependence of the expected number of nodal domains

on the spectral parameter was studied. The numerically observed patterns were accurately reproduced by assuming a random waves model where the distribution of the eigenvectors converges to that of a Gaussian random field on the regular tree  $T_d$ .

Here we study further the morphology of level sets, and in particular investigate the percolation transition observed numerically at a critical level  $\alpha_c$  which depends on the eigenvalue under consideration. Invoking again the random wave hypothesis we are able to reproduce the numerical simulations, and thus bring further supporting evidence for the applicability of the random waves model. In this way we provide the discrete analogue to Bogomolny's percolation hypothesis [2], and shed more light on the surprising connection between percolation and spectral theory. While for two dimensional billiards, if a critical level set exist it must be the nodal set by duality arguments (e.g. [13]), the fact that in the present case the percolation threshold typically occurs at  $\alpha_c \neq 0$  should not come as a surprise. This is consistent with classical results about probabilistic percolation on graphs [14].

The paper is organized in the following way. After some necessary preliminaries and a review of some pertinent results, we describe the results of numerical simulations where the distribution of the largest  $\alpha$ -level set for different eigenvectors of random regular graphs were studied. The numerically deduced critical level  $\alpha_c(\lambda, d)$  as a function of the eigenvalue and the degree of the graph summarizes these computation. The next chapter is dedicated to a systematic construction of a Gaussian random waves model on the  $d$ -regular tree [15]. For this process, we are able to prove that a percolation transition occurs. We then show that the critical level sets computed for  $T_d$  reproduce accurately the numerical simulations for  $d$  regular graphs, which is reasonable in light of the local tree property. Yet, a rigorous proof of this observation is lacking. The paper is concluded by a comparison between the observed critical behavior and the percolation model for two dimensional billiards.

## 2. Preliminaries

### 2.1. Elementary definitions

A graph  $G$  is a discrete set of vertices, connected by edges. We denote the size of a graph by  $|G| = n$ , where by an abuse of notation, we use the symbol  $G$  to denote both the graph and its set of vertices. We will consider only simple graphs, i.e. graphs containing no loops or multiple edges. A graph  $G$  is *d-regular* if for every vertex  $v \in G$ , the degree of  $v$  (or the number of edges connected to  $v$ ) is exactly  $d$ . For vertices  $v, v' \in G$ , we define the graph distance  $|v - v'|$  as the length of the shortest walk in  $G$  from  $v$  to  $v'$ .

A graph is completely specified by its adjacency operator  $A$ , where  $A_{ij} = 1$  if  $v_i, v_j \in G$  are adjacent vertices, or zero otherwise. As the adjacency operator is real and symmetric, it has  $n$  real eigenvalues. We will denote the spectrum and eigenvectors of the adjacency operator by

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \quad (A - \lambda_j)f_j = 0$$

The eigenvectors are normalized through the paper according to the convention

$$\sum_{v \in G} f_j^2(v) = n$$

so that  $\text{Var}(f(v))$  does not vanish as  $n \rightarrow \infty$ .

### 2.2. $G(n, d)$ - the ensemble of random regular graphs

For a given value of  $n, d \in \mathbb{N}$ , the ensemble  $G(n, d)$  consists of all  $d$  regular graphs on  $n$  vertices, equipped with the uniform measure.

The geometrical and spectral properties of  $G(n, d)$  have been extensively studied for more than 30 years, and are successfully applied in various fields such as combinatorics, information theory, pseudo-randomness and more (see [16] for a review). In the following we would like to investigate the properties of the eigenvectors of a typical  $(n, d)$  graph, for a fixed  $d \geq 3$  in the limit  $n \rightarrow \infty$ . Here and in the following, by stating that a typical  $(n, d)$  graph has a property  $\mathcal{X}$ , we mean that the probability that a graph  $G \in G(n, d)$  has the property  $\mathcal{X}$  converges to one as  $n \rightarrow \infty$ .

In [17] the distribution of short cycles in  $G(n, d)$  is calculated. Denoting by  $C_k$  the number of independent  $k$ -cycles in a graph, it is shown that the distribution of  $\{C_k\}$ ,  $3 \leq k \leq \log_{d-1} n/2$  converges as  $n \rightarrow \infty$  into independent Poisson random variables with an expectation value

$$\mathbb{E}(C_k) = \frac{(d-1)^k}{2k}. \quad (2.1)$$

Since the expected number of  $k$ -cycles does not increase with the size of the graph, we find that a ball of radius  $c \log n$  around a random vertex has a tree structure with probability  $1 - n^{c-1}$ , for every  $c < 1/2$ . As a result, the local structure of an  $(n, d)$  graph near most of its vertices is identical to that of  $T_d$  - the  $d$ -regular (infinite) tree.

The diameter of an  $(n, d)$  graph, i.e. the maximal distance between vertices in  $G$ , is given by [18]

$$\text{diam}(G) = \log(n \log n) + O(1) \quad (2.2)$$

(here and in what follows the logarithm base is  $(d-1)$ ). This result shows that the typical distance between vertices along the boundary of the 'local tree' is of the same magnitude as the distance between two arbitrary vertices in  $G$ .

The local resemblance between  $(n, d)$  graphs and  $T_d$  is reflected in the spectral density of their adjacency operator as well. Both the spectrum of the tree [19] and the limiting spectral density of  $G(n, d)$  [20] are supported on the interval

$$\sigma(T_d) = [-2\sqrt{d-1}, 2\sqrt{d-1}] \quad (2.3)$$

with a spectral density, given by

$$p(\lambda) = \frac{d}{2\pi} \frac{\sqrt{4(d-1) - \lambda^2}}{d^2 - \lambda^2} \quad (2.4)$$

For every connected regular graph which is not bipartite, the unique stationary distribution for random walks on the graph is uniform over the vertices. The rate of convergence into the stationary distribution is dictated by  $P_{v,v'}^{(k)}$  - the probability that a walk of length  $k$  which begins at the vertex  $v$  will terminate in  $v'$ . In [7] it is shown that for a typical  $(n, d)$  graph

$$\limsup_{k \rightarrow \infty} \left| P_{v,v'}^{(k)} - \frac{1}{n} \right|^{1/k} = e^{-\gamma} \quad (2.5)$$

where the Lyapunov exponent

$$\gamma = 1 - \frac{2\sqrt{d-1}}{d} + O\left(\frac{1}{\log n}\right)$$

is strictly positive for  $d \geq 3$ . As a result, random walks on a typical  $(n, d)$  graph are exponentially mixing with a Lyapunov exponent  $\gamma \approx 1 - 2\sqrt{d-1}/d$ . This observation justifies attributing the title “chaotic” to  $(n, d)$  graphs, as was done in the introduction section. For further detail on expanding (equivalently, mixing) graphs and the relations between their spectral, geometrical and dynamical properties we refer the reader to [16].

### 2.3. The random waves model for the adjacency eigenvectors of $G \in G(n, d)$

The main tool in the present study is the random wave model [1, 15]. It is based on the observation that for the  $d$  regular tree  $T_d$ , the distribution of a typical eigenvector can be approximated by a real Gaussian process  $\mathcal{GS}_d(\lambda)$ . The process associates random functions  $\psi_\omega : T_d \rightarrow \mathbb{R}$  to the regular tree, so that for every subset of vertices  $\{v_j\} \subset T_d$ , the distribution of any linear combination of  $\psi_\omega(v_j)$  is Gaussian.

A Gaussian process is uniquely characterized by its mean and covariance operator. Introducing the Chebyshev Polynomials of the second kind

$$U_k(x) = \frac{\sin((k+1)\cos^{-1}(x))}{\sin(\cos^{-1}(x))} \quad (2.6)$$

and the polynomials [19, 21]

$$\phi^{(\lambda)}(k) = (d-1)^{-k/2} \left( \frac{d-1}{d} U_k\left(\frac{\lambda}{2\sqrt{d-1}}\right) - \frac{1}{d} U_{k-2}\left(\frac{\lambda}{2\sqrt{d-1}}\right) \right) \quad (2.7)$$

It was shown in [1, 15] that for every  $\lambda \in \sigma(T_d)$  (2.3), the random Gaussian process  $\mathcal{GS}_d(\lambda)$  which is characterized by the covariance

$$\text{Cov}(\psi_\omega(v), \psi_\omega(v')) = \phi^{(\lambda)}(|v - v'|) \quad (2.8)$$

Has the following properties:

- (i)  $(A_{T_d} - \lambda I)\psi_\omega = 0$  for almost every  $\psi_\omega \in \mathcal{GS}_d(\lambda)$ .
- (ii)  $\mathcal{GS}_d(\lambda)$  is invariant with respect to the symmetries of  $T_d$ .
- (iii) The process is normalized, so that  $\text{Var}(\psi_\omega(v)) = 1$ .

The random wave model is based on the conjecture that the distribution of a typical adjacency eigenvector of a graph  $G \in G(n, d)$  graph, with an eigenvalue  $\lambda \in \sigma(T_d)$  converges locally to that of  $\mathcal{GS}_d(\lambda)$ . This hypothesis, was supported by various numerical tests, and found a partial formal justification in [15]. The random waves hypothesis will be the basis for the analysis of the morphology of level sets in  $(n, d)$  graphs, which will be carried out in this paper. It is the analogue of Berry's model for the distribution of eigenfunctions for chaotic billiards [22].

### 3. Level sets percolation on $(n, d)$ graphs

In this section we shall present the numerical evidence which led us to propose that level sets undergo a percolation transition in the limit  $n \rightarrow \infty$ .

For a graph  $G$ , a real function  $f(G)$  and a given  $\alpha \in \mathbb{R}$ , we denote by

$$\tilde{G}_\alpha(f) = \{v \in G, f(v) > \alpha\} \quad (3.1)$$

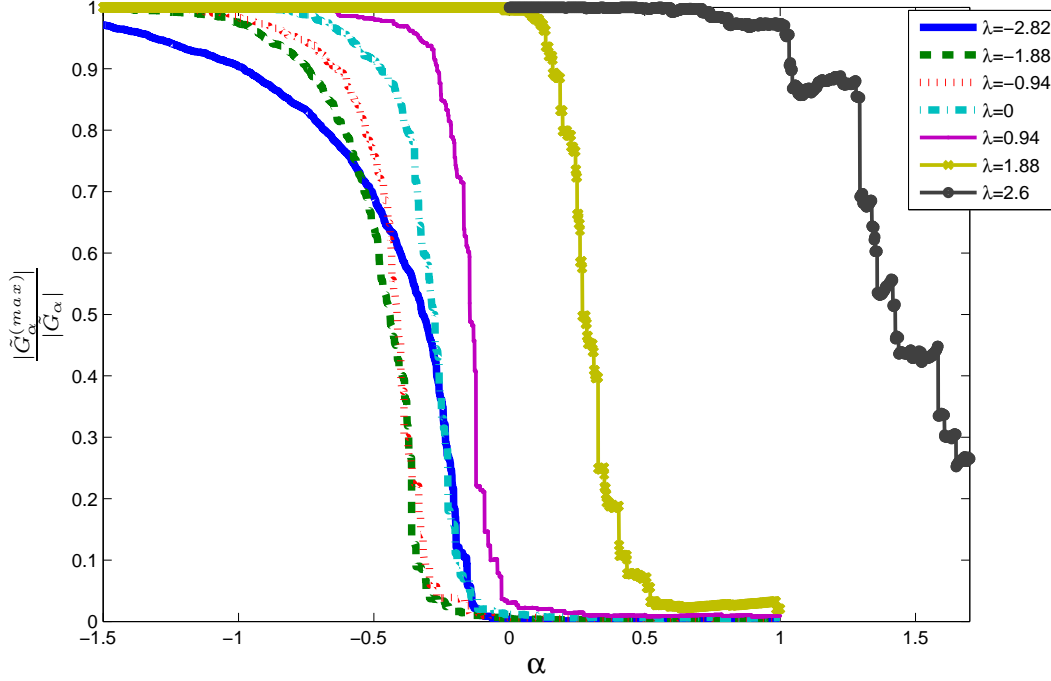
the induced graph, which is obtained by deleting all the vertices for which  $f(v)$  is below the threshold  $\alpha$ ; The  $\alpha$ -level sets of  $f$  in  $G$  are the connected components of  $\tilde{G}_\alpha(f)$ .

Since there is no known analytical expression for the distribution of eigenvectors in an  $(n, d)$  graph, we cannot offer an expression for the limiting distribution of the  $\alpha$ -level sets for this ensemble. However, motivated by the resemblance between the spectral and eigenfunctions statistics for  $G(n, d)$  and chaotic billiards, we have looked for a numerical evidence to a phase transition in  $(n, d)$  level sets.

For a given graph  $G \in G(n, d)$ ,  $\alpha \in \mathbb{R}$  and an eigenvector  $(A - \lambda I)f = 0$ , we define  $\tilde{G}_\alpha^{(max)}$  to be the largest component of  $\tilde{G}_\alpha(f)$  (3.1) and evaluate the ratio  $|\tilde{G}_\alpha^{(max)}|/|\tilde{G}_\alpha|$ . We have generated (following [23]) and diagonalized (using MatLab) random regular graphs on up to 4000 vertices with degrees ranging from 3 to 15. For each eigenvector  $f$  of each graph we have measured the ratio  $|\tilde{G}_\alpha^{(max)}|/|\tilde{G}_\alpha|$  while varying  $\alpha$  from  $\min(f)$  to  $\max(f)$ . The ratios  $|\tilde{G}_\alpha|/|\tilde{G}_\alpha^{(max)}|$  are plotted in figure 1 as a function of  $\alpha$  for a 3-regular graph on 4000 vertices. The different lines correspond to different values of  $\lambda$  are given

in the inset.

A sharp transition in the normalized size of the level sets is evident: for every value



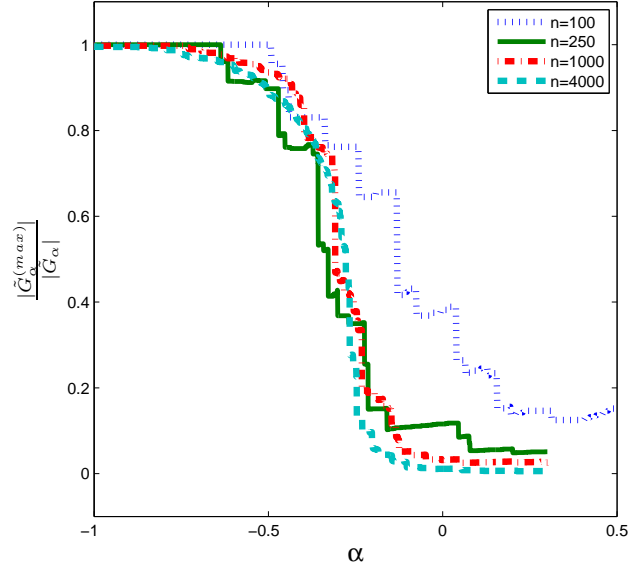
**Figure 1.** The ratio between the magnitude of the largest level set to the size of the induced graph  $\tilde{G}_\alpha$  for a single realization of a  $(4000, 3)$  graph. Each curve corresponds to one eigenvector, whose level sets are interrogated by increasing  $\alpha$ .

of  $\lambda$  there is a narrow window in the vicinity of some  $\alpha_c(\lambda, d)$ , so that for  $\alpha > \alpha_c$ , the ratio  $|\tilde{G}_\alpha^{(max)}|/|\tilde{G}_\alpha|$  is close to zero, while for  $\alpha < \alpha_c$ ,  $|\tilde{G}_\alpha^{(max)}|/|\tilde{G}_\alpha|$  is of order one. The described phenomenon was observed for all the tested values of  $d$  and for all the examined eigenvectors.

Moreover, repeating the experiment, while varying the size of the graph, we have observed that the value of  $\alpha_c(\lambda, d)$  does not vary with  $n$ , while the transition becomes sharper as  $n$  increase. As an example, in figure 2 we plot the variation in  $|\tilde{G}_\alpha^{(max)}|/|\tilde{G}_\alpha|$  for the eigenvectors which correspond to  $\lambda \approx 0$ , for different realizations of 3-regular graphs of various sizes.

This extensive corpus of numerical data provides strong evidence supporting the



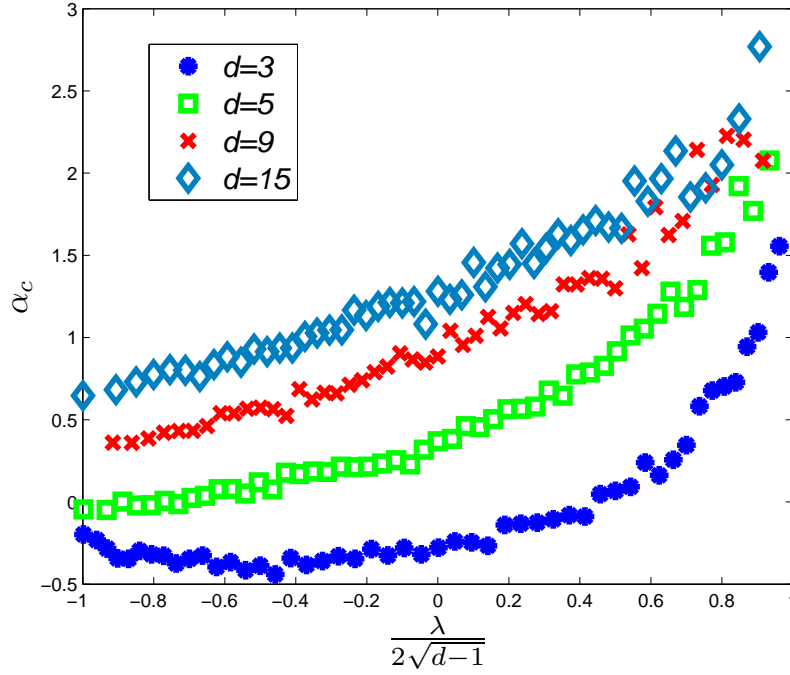


**Figure 2.** A comparison of the curve  $|\tilde{G}_\alpha^{(max)}|/|\tilde{G}_\alpha|$ , for several 3-regular graphs of varying size. the curves correspond to  $n = 100, 250, 1000, 4000$  where in each graph we consider the eigenvector with the closest eigenvalue to zero.

existence of a phase transition for the level sets of  $(n, d)$  eigenvectors. Namely, for every  $d \geq 3$  and  $\lambda \in \sigma(T_d)$ , there may exist an  $\alpha_c(\lambda, d) \in \mathbb{R}$ , so that the level sets of a typical  $(n, d)$  eigenvector which corresponds to the eigenvalue  $\lambda$  are all microscopic for  $\alpha > \alpha_c$ , while for  $\alpha < \alpha_c$  a macroscopic component is expected to appear.

Note that the suggested transition differs from the percolation hypothesis for chaotic billiards in two main aspects. First, while for billiards the transition is expected to follow the characteristics of non-correlated percolation, for  $(n, d)$  graphs we expect correlations to be relevant (as will be discussed in section 5). Second, Unlike the percolation model for billiards, the critical threshold for regular graphs depends on the corresponding eigenvalue.

In order to estimate the dependence of  $\alpha_c(\lambda, d)$  on its arguments, we have chosen (somewhat arbitrarily) to identify  $\alpha_c(\lambda, d)$  with the steepest point of the curve  $|\tilde{G}_\alpha^{(max)}|/|\tilde{G}_\alpha|$ , obtained for a graph of size  $n = 4000$ . In figure 3 we present our numerical estimate of the critical curves. The results suggest that for a given value of  $d$ , the critical



**Figure 3.** A numerical estimation to  $\alpha_c(\lambda, d)$  for  $d = 3, 5, 6, 12$ .

threshold increases monotonically with  $\lambda$  for all  $d > 3$ . For the lowest degree,  $d = 3$ , the critical curve  $\alpha_c(\lambda, d = 3)$  shows a minimum in the vicinity of  $\lambda \approx -0.52$ .

In the next sections we shall show that the numerical results summarized above can be reproduced theoretically by considering first the level sets in the random process  $\mathcal{GS}_d(\lambda)$  on  $T_d$ , and then *assuming* the validity of the random waves conjecture for the adjacency eigenvectors in  $G(n, d)$ .

#### 4. The distribution of level sets in $\mathcal{GS}_d(\lambda)$

In this section we shall study the distribution of the level sets in  $GS$  and prove that they undergo a percolation transition, for which the critical threshold  $\alpha_c(\lambda, d)$  can be computed.

As the process  $\mathcal{GS}_d(\lambda)$  is Gaussian and characterized by the covariance operator (2.8), it is possible to rigorously analyze its level sets statistics. Setting

$$T_\omega(\alpha) = \{v \in T_d, \psi_\omega(v) > \alpha\} \quad (4.1)$$

to denote the induced  $\alpha$  level sets tree for a given  $\psi_\omega \in \mathcal{GS}_d(\lambda)$ , we show that:

**Theorem 4.1.**  *$\forall \lambda \in \sigma(T_d)$ , there exists an  $\alpha_c \in \mathbb{R}$  so that for almost every realization  $\psi_\omega \in \mathcal{GS}_d(\lambda)$ ,  $T_\omega(\alpha)$  has an infinite component for  $\alpha < \alpha_c$ , but only finite components for  $\alpha > \alpha_c$ .*

As the proof of the theorem is rather technical, we refer the interested reader to [15] where a complete and detailed proof of the theorem can be found. Here, we shall provide the main line of the proof, skipping much of the technical aspects.

It is important to note that for a given  $\alpha \in \mathbb{R}$ , the  $\alpha$ -level sets of the process  $\mathcal{GS}_d(\lambda)$  is a homogeneous vertex process on  $T_d$ . That is, the probability measure of the process is invariant with respect to the symmetries of  $T_d$ .

For any homogenous process, the probability that  $v, v' \in T_d$  belongs to the same connected component depends only on the distance between the vertices and will be denoted by  $P_{|v-v'|}$ . Denoting the sphere of radius  $k$  around  $v \in T_d$  by

$$\Lambda_k(v) = \{v' \in T_d, |v - v'| = k\} \quad (4.2)$$

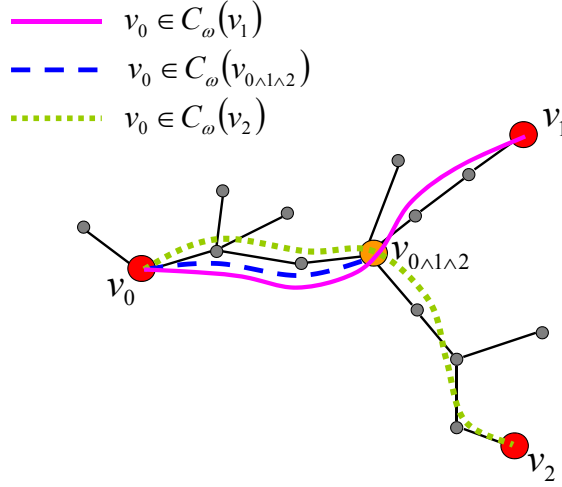
we find that the probability that  $v$  is connected to its  $k$ -sphere is at most  $|\Lambda_k(v)|P_k$ . Since  $|\Lambda_k| = d(d-1)^k$ , we obtain that if

$$\limsup_{k \rightarrow \infty} P_k^{1/k} < \frac{1}{d-1} \quad (4.3)$$

then the probability that the connected component of  $v$  exceeds the radius  $k$  decays exponentially with  $k$ , so that the probability to find an infinite component is zero and the process is subcritical.

We should note that the opposite statement is not necessarily correct, i.e. there are percolation processes on  $T_d$  for which  $\liminf_{k \rightarrow \infty} P_k^{1/k} > 1/(d-1)$ , but do not contain infinite components, due to long range correlations.

The influence of the correlations which are induced by the process can be formally evaluated according to the next classification of random processes on trees, introduced in [24].



**Figure 4.** For a quasi-bernoulli process, the probability to find  $v_0 \in C_\omega^\alpha(v_1)$  (i.e. that the continuous purple line is entirely occupied in  $\Gamma_\omega$ ) conditioned that  $v_0 \in C_\omega^\alpha(v_2)$  (dotted green line is occupied) is uniformly bounded by the probability that  $v_0 \in C_\omega^\alpha(v_1)$ , conditioned that  $v_0 \in C_\omega^\alpha(v_0 \wedge v_1 \wedge v_2)$  (dashed blue line is occupied). Therefore the effect of values which are obtained by the process along the path  $v_{0 \wedge 1 \wedge 2} \rightarrow v_2$  has only a limited effect on the path  $v_{0 \wedge 1 \wedge 2} \rightarrow v_1$ .

**Definition 4.2.** A random percolation process on a tree graph  $\Gamma$ , associating for all  $\omega \in \Omega$  an induced subgraph  $\Gamma_\omega \subset \Gamma$ , is a *quasi Bernoulli* process, if  $\exists M < \infty$ , such that  $\forall v_0, v_1, v_2 \in \Gamma$ :

$$\frac{\mathbb{P}(v_1 \in C_{\Gamma_\omega}(v_0) | v_2 \in C_{\Gamma_\omega}(v_0))}{\mathbb{P}(v_1 \in C_{\Gamma_\omega}(v_0) | v_{0 \wedge 1 \wedge 2} \in C_{\Gamma_\omega}(v_0))} \leq M. \quad (4.4)$$

where  $v_{0 \wedge 1 \wedge 2}$  is the intersection of the simple paths in  $\Gamma$  between the three vertices (see figure 4) and  $C_{\Gamma_\omega}(v_0)$  is the connected component of  $v_0$  in  $\Gamma_\omega$ .

According to definition 4.2, in a quasi Bernoulli process, the distribution of the process along any simple path is only moderately affected by the realization of the process outside the path. As an example, it can be verified that a Markovian process is quasi Bernoulli.

The quasi Bernoulli classification provides a simple criterion for the relevance of the long range correlations of the process on its macroscopic properties, as suggested by the

next lemma [24]‡:

**Lemma 4.3. (*Lyons*)** *Let  $\{\Omega, \mathbb{P}\}$  be an invariant quasi Bernoulli process on  $T_d$ , which associates  $\forall \omega \in \Omega$  an induced graph  $T_\omega \subset T_d$ . If §*

$$\lim_{k \rightarrow \infty} P_k^{1/k} < \frac{1}{d-1} \quad (4.5)$$

*then, all the connected components of  $T_\omega$  are almost surely finite. If*

$$\lim_{k \rightarrow \infty} P_k^{1/k} > \frac{1}{d-1} \quad (4.6)$$

*$T_\omega$  will almost surely have a component of an infinite cardinality.*  $\square$

An equivalent phrasing of the lemma is the following: for a quasi Bernoulli process on  $T_d$ , all the components are almost surely finite if their expected cardinality is finite, while if the expected cardinality diverges, infinite components will almost surely exist. As the  $\alpha$ -level sets are monotonically decreasing in  $\alpha$ , we find that in order to prove theorem 4.1 it is enough to show that

- For every  $\lambda \in \sigma(T_d)$  and  $\alpha \in \mathbb{R}$ , the  $\alpha$ -level sets of  $\mathcal{GS}_d(\lambda)$  are quasi Bernoulli
- For small enough values of  $\alpha$  there almost surely exist infinite level sets, while for large enough values the  $\alpha$ -level sets are all finite with probability one.

The first condition guaranties that for every  $\lambda \in \sigma(T_d)$  and  $\alpha \in \mathbb{R}$ , the probability to find an infinite component is either zero or one (according to lemma 4.3). Assuming the level sets are quasi Bernoulli, and since  $\lim_{k \rightarrow \infty} P_k^{1/k}(\alpha)$  is monotone in  $\alpha$ , the second condition ensures that for every  $\lambda \in \mathbb{R}$  a supercritical and a subcritical phases exist, where the transition between the two occurs at  $\alpha_c(\lambda, d)$  which is given by the implicit expression

$$\lim_{k \rightarrow \infty} P_k^{1/k}(\alpha_c) = \frac{1}{d-1} \quad (4.7)$$

‡ The theorem as it appears in [24] characterizes general quasi-Bernoulli process on an arbitrary tree graph. For the sake of clarity, we consider here only the restriction of the theorem to invariant percolation processes on  $T_d$ .

§ Note that for a quasi Bernoulli process, the limit  $\lim_{|v'-v| \rightarrow \infty} (\mathbb{P}(v' \in C_{T_\omega}(v)))^{1/|v-v'|}$  does exists. This can be verified by restricting the condition (4.4) to cases in which  $v_0$  is on the simple path between  $v_1$  to  $v_2$ , i.e.  $v_0 = v_0 \wedge 1 \wedge 2$ .

To verify the existence of a subcritical regime, we note that if a simple path  $U$  of length  $k$  is contained in an  $\alpha$ -level set, then necessarily  $\forall v_j \in U, \psi_\omega(v_j) > \alpha$ . Therefore, setting  $\Psi_\omega(U) = \sum_{j=1}^k \psi_\omega(v_j)$ , we obtain that

$$\begin{aligned} P_k(\alpha) &= \mathbb{P}(\forall v_j \in U, \psi_\omega(v_j) > \alpha) \\ &< \mathbb{P}(\Psi_\omega(U) > k\alpha) \end{aligned} \quad (4.8)$$

Note that  $\Psi_\omega(U)$  is a Gaussian random variable, with mean zero and variance

$$\begin{aligned} \text{Var}(\Psi_\omega(U)) &= \mathbb{E} \left[ \left( \sum_{ij} \psi_\omega(v_i) \psi_\omega(v_j) \right)^2 \right] \\ &= k \left( \phi^{(\lambda)}(0) + 2 \sum_{j=1}^{k-1} \frac{k-j}{k} \phi^{(\lambda)}(j) \right) < k\Phi^{(\lambda)} \end{aligned} \quad (4.9)$$

where  $\Phi^{(\lambda)} = \phi^{(\lambda)}(0) + 2 \sum_{j=1}^{\infty} |\phi^{(\lambda)}(j)|$  (2.7).

Since  $\phi^{(\lambda)}(j)$  is exponentially decreasing in  $j$  (2.7), we find that  $\Phi^{(\lambda)} < \infty$ . Therefore

$$P_k(\alpha) < \frac{1}{\sqrt{2\pi k\Phi^{(\lambda)}}} \int_{k\alpha}^{\infty} \exp\left(-\frac{x^2}{2k\Phi^{(\lambda)}}\right) < e^{-\beta\alpha^2 k} \quad (4.10)$$

where  $\beta = (2\Phi^{(\lambda)})^{-1}$ .

As  $|\Lambda_k| = d(d-1)^{k-1}$ , we obtain from (4.3) and (4.10) that the  $\alpha$ -level sets are subcritical for  $\alpha > \sqrt{\ln(d-1)/\beta}$ .

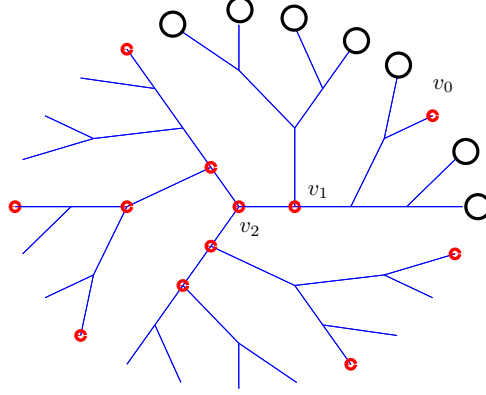
The existence of infinite  $\alpha$ -level sets for small enough  $\alpha$  is a straight-forward consequence of [25], where it is shown that every homogenous vertex percolation process on  $T_d$ , for which the survival probability exceeds  $d/2(d-1)$  is supercritical. As  $\psi_\omega(v)$  distributes as a normal variable with mean zero and variance one, the vertex survival probability for  $T_\alpha(\psi_\omega)$  is given by

$$P_0(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-x^2/2} . \quad (4.11)$$

Since  $d/2(d-1) \leq 3/4$  for every  $d \geq 3$  and as according to (4.11)  $P_0(\alpha) > 3/4$  for  $\alpha \leq -0.68$ , we obtain that for every  $d \geq 3$  and  $\lambda \in \sigma(T_d)$ , the  $\alpha$ -level sets of  $\mathcal{GS}_d(\lambda)$  are supercritical for  $\alpha < -0.68$ .

In order to prove that the  $\alpha$ -level sets of  $\mathcal{GS}_d(\lambda)$  are quasi Bernoulli, we have to show that the long range correlations do not dominate the structure of the random tree

$T_\omega(\alpha)$  (4.1). A major step toward this goal is the next theorem, which identifies the following Markov property of the underlying process  $\mathcal{GS}_d(\lambda)$ :



**Figure 5.** A set of vertices which corresponds to the requirements of theorem 4.4.  $v_0, v_1$  and  $v_2$  are denoted explicitly;  $\{v_j\}_{j \geq 3}$  are marked by full red circles;  $\{v' \in \Lambda_k, |v' - v_1| < |v' - v_2|\}$  are marked by hollow black circles (see 4.12) for the case  $k = 3$ .

**Theorem 4.4.** Let  $\{v_j\}_{j=0}^k \subset T_d$  (for  $k \geq 3$ ), so that  $|v_1 - v_2| = 1$ , and  $v_1$  is on the simple path between  $v_0$  to  $\{v_j\}_{j=2}^k$  (see figure 5). Then  $\forall \lambda \in \sigma(T_d)$ , the distribution of  $\psi_\omega(v_0)$  conditioned on  $\{\psi_\omega(v_j)\}_{j=1}^k$  is independent of  $\{\psi_\omega(v_j)\}_{j=3}^k$

The proof of theorem 4.4 relies on the observation that for any adjacency eigenvector  $\psi : T_d \rightarrow \mathbb{R}$ , the value of  $\psi(v_1), \psi(v_2)$  determines the value of the sum

$$\sum_{v' \in \Lambda_k(v_1), |v' - v_1| < |v' - v_2|} \psi(v') \quad (4.12)$$

(see figure 5). Since the distribution of  $\mathcal{GS}_d(\lambda)$  is invariant with respect to the symmetries of  $T_d$ , we obtain from the last observation that given  $\{\psi_\omega(v_j)\}_{j=1}^k$ , the expected value of  $\psi_\omega(v_0)$  is determined by  $\psi_\omega(v_1), \psi_\omega(v_2)$ .

Since non-correlated components of a multi-normal random vector are also independent (e.g. [26]), the theorem follows.  $\square$

The described dominance of short range correlations in the process  $\mathcal{GS}_d(\lambda)$ , is utilized in [15], to bound rigorously the effect of long range correlation in  $T_\omega(\alpha)$ , resulting in the identification of  $T_\omega(\alpha)$  as a quasi Bernoulli process and in the establishment of theorem 4.1.

The last step in the analysis is obtained by solving (4.7) for various values of  $d$  and  $\lambda \in \sigma(T_d)$ . The results are shown in figure 6, superimposed on the numerical data obtained for  $G(n, d)$  graphs and discussed in the previous section (figure 3). The agreement between the  $G(n, d)$  data and the critical threshold levels computed for the  $\mathcal{GS}_d(\lambda)$  process on  $T_d$  is perfect (including in particular the non monotonic behavior of  $\alpha_c(\lambda, d = 3)$  which is reproduced as well). It strongly supports the random waves conjecture for  $G(n, d)$ . As was already stated above, the random wave conjecture is valid locally. In the present context, however it applies globally. The proof of the random wave conjecture is still lacking and is a challenge for experts in probabilistic graph theory.

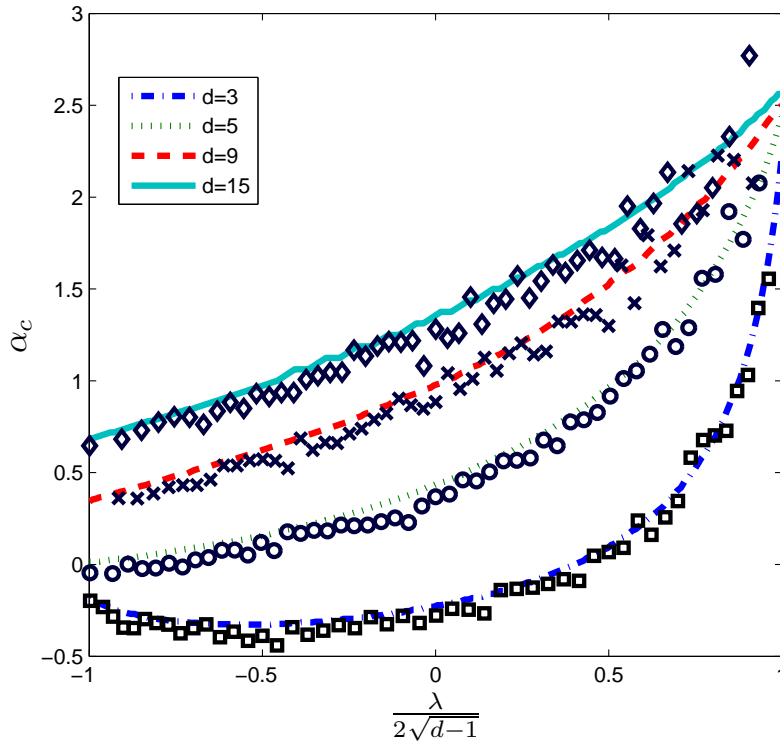
## 5. Discussion

The identification of a phase transition for the level sets of  $\mathcal{GS}_d(\lambda)$  (and the observed transition for  $G(n, d)$  eigenvectors) may be considered as an extension of the percolation hypothesis for random waves on  $\mathbb{R}^2$  and for chaotic billiards [2]. We would like to conclude this paper by a short comparison between the two models.

We should note that the formal derivation for both of the model is based on the statistical properties of the corresponding random waves model. while the heuristic justification of the percolation hypothesis for two-dimensional random waves is based on arguments which are not clearly justified [27, 28, 29, 12], theorem 4.1 provides a rigorous proof to the existence of a critical level-set for  $\mathcal{GS}_d(\lambda)$ .

In both of the cases, the applicability of the suggested model to the corresponding chaotic system (a billiard or a mixing regular graph) is a consequence of the corresponding random waves model and not of an independent derivation. However the consistency of





**Figure 6.** A comparison between the value of  $\alpha_c(\lambda, d)$  for the Gaussian process  $\mathcal{GS}_d(\lambda)$ , given by equation 4.7 (lines) to the numerical estimation of  $\alpha_c(\lambda, d)$  for  $(n, d)$  graphs (markers).

the models with various numerical tests, such as [9, 2, 30, 11] (for billiards) and [1] (for  $(n, d)$  graphs) provide a firm support both for the percolation and the random-waves models.

The arguments behind the two percolation models differ significantly. The percolation hypothesis for two dimensional chaotic billiards is justified in [2, 10] by dimensional arguments which prevents its generalization for generic chaotic systems, or even billiards of higher dimensions. In particular it is based on the topological identification of a random wave with the square lattice [2, 12, 28] and the particular critical threshold  $p_c = 1/2$  for self-dual bond percolation processes on the square lattice. In addition, the neglect of correlation is based on a careful application of the Harris criterion [31], which is valid only for two-dimensional systems [10, 28].

Unfortunately, the identification of a critical level set for the process  $\mathcal{GS}_d(\lambda)$  demonstrates the same weakness, as it utilizes repeatedly the tree structure of  $T_d$ . Nevertheless, the identification of a critical level sets for the two seemingly non-related waves ensembles hints on a universal mechanism behind the phenomena. In addition the dependence of the critical point (4.7) in the spectral parameter  $\lambda$ , implies that the suggested transition for the eigenfunctions of a generic chaotic system (assuming it indeed exists) may exhibit a more complicated behavior then the simple model which is suggested in [2].

## Acknowledgments

The authors would like to thank I. Benjamini, M. Aizenman and O. Zeitouni for enlightening discussions, comments and suggestions.

The work was supported by the Minerva Center for non-linear Physics, the Einstein (Minerva) Center at the Weizmann Institute and the Wales Institute of Mathematical and Computational Sciences) (WIMCS). Grants from EPSRC (grant EP/G021287), ISF (grant 166/09), BSF (710021/1) and Afeka college of engineering are acknowledged.

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